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Some Relationships Between Representation Systems and Physics. I: Representation Systems

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Abstract

A representation system developed by Smullyan is discussed briefly. Additional notation is introduced to make it suitable for problems concerned with physical systems. Rules for the formation of a concatenation operation called the product, are introduced.

Axiomatisation of a physical theory is difficult. To begin with the requirement that every relevant semantic assumption should be related closely to logical formalism is likely to result, finally, in an extreme form of operationalism of the variety devised by Eddington (1940), which, though it may possess considerable beauty and mathematical rigour, involves the problem that it is hard to link satisfactorily with well-known empirical facts in physical science. An opposite viewpoint, still adopted by many, is that one need not bother to axiomatise rigorously at all. It is noteworthy, however, that when real difficulties arise, as in relativistic quantum field theory, physicists sometimes tend to return to axiomatic methods. The desirability of axiomatics is well emphasised elsewhere (Bunge, 1967).

The present work steers a middle course, invoking a partly semantic and partly formal presentation to describe a theory (in this case non-relativistic quantum mechanics) and then discussing some of the semantics in terms of logical structures. One such structure is described in some detail in this paper, this structure and its modifications should be adaptable to various problems which involve the change of state of a system. The crucial point, of course, is to be sure that one is specifying concepts implicit in an essential way in the part of the theory hitherto dealt with discursively; ultimately this would be proved using experimental tests.

The second of these two papers relates a representation system to descriptions of real and unreal physical processes. The only processes which can be referred to by provable sentences in the representation system are the real processes, and sentences referring to unreal processes are refutable. The Hilbert spaces then needed for description of non-relativistic quantum mechanics are superspaces of those usually used, and an absolute timescale is required.

For convenience Cole's abbreviations for the terms 'axiom', 'definition', 'remark', 'theorem', 'convention', 'proof', and 'example' are used (Cole, 1968). The mathematical structure used is developed from the representation theory of Smullyan (1961), some knowledge of which is desirable for an understanding of this work. The particular feature of the present system is the product as defined in DF(14).

RMK/DF(1): First the contents of the representation system Z are given by AX(1) to AX(5), and the properties of Z are summarised in DF(1) to DF(11).

AX(1): A denumerable set of expressions E, with a one-to-one Godel numbering g of E onto N. N is a subset of the positive integers.

AX(2): A subset of E, called S, containing sentences of Z.

AX(3): Subsets T and R of S.

RMK(2): T and R are subsequently regarded as the sets of the true and refutable sentences respectively.

AX(4): A subset P of E called predicates.

AX(5): A mapping M from $E \times N$ into E. M assigns to each expression X and positive integer n a unique expression $M(X,n)$, with the following property:

$$X \in P,\ n \in N \rightarrow M(X,n) \in S$$

DF(1): If $H \in P$, H represents a number set A iff for each $n \in N$,

$$n \in A \leftrightarrow M(H,n) \in T$$

DF(2): For each $H \in E$ define $g(H) = h$.

CVN(1): Lower-case letters are the Godel numbers of the corresponding upper-case letters if the latter are expressions in a representation system.

DF(3): If $h \in N$, the formation of $M(H,h)$ is called diagonalisation, $M(H,h)$ is called a diagonal sentence iff $H \in P$.

DF(4): For a set, W, of expressions, define the set W^* of numbers by the condition

$$i \in W^* \leftrightarrow M(I,i) \in W \qquad (I \in E)$$

CVN(2): W^* is then the set of Godel numbers of expressions diagonalisable in W, and the asterisk applied to a symbol for a set has such a meaning throughout this work.

DF(5): A sentence X is known as a Godel sentence for a set W of expressions iff

$$X \in T \leftrightarrow X \in W$$

DF(6): X is a Godel sentence for a set A of numbers iff

$$X \in T \leftrightarrow g(X) \in A$$

TH(1): If H represents W^* in Z, $M(H,h)$ is a Godel sentence for the set W.
RMK(3): Make additions to the notation as described by CVN(3), CVN(4), and CVN(5). These do not alter the properties of Z.
CVN(3): Iff $g(Y) = y$ then $g^{-1}(y) = Y$ each $y \in E$

$$M(H,h) = M(g^{-1}(h), n) = G(h,n) \qquad \text{each} \qquad h, n \in N$$

CVN(4): $g(G(h,n)) = g(h,n)$ each h, $n \in N$.
CVN(5): Let ϕ, Φ be distinct objects not in $N \cup E$. Then let $X \subseteq N$ define the function P_x on N by

$$P_x(y) = y \qquad \text{if} \qquad y \in X$$
$$P_x(y) = \phi \qquad \text{if} \qquad y \notin X$$

DF(7): g can be extended to a one-to-one function g_1 mapping $E \cup \{\Phi\}$ onto $N \cup \{\phi\}$ by defining

$$g_1(W) = g(W) \qquad \text{if} \qquad W \in E$$
$$g_1(W) = \phi \qquad \text{if} \qquad W = \Phi$$

DF(8): $G(a,b)$ is extended to $G_1(a,b)$ by

$$G_1(a,b) = G(a,b) \qquad \text{if} \qquad a \neq \phi \qquad \text{and} \qquad b \neq \phi$$
$$G_1(a,b) = \phi \qquad \text{if} \qquad a = \phi \qquad \text{or} \qquad b = \phi \qquad \text{or} \qquad a = b = \phi$$

RMK(4): Representation of number set A by H can now be written as follows

$$G(h,n) \in T \leftrightarrow G_1(h, P_A(n)) \in E$$

Also if $G_1(h, P_{W^*}(n)) \in E$ iff $G(h,n) \in T$ then $G(h,h)$ is a Godel sentence over W.

One now has a formalism whose decision properties are very simple, a fact exemplified by Smullyan's version of Godel's theorem.
DF(9): Dot multiplication is used as follows.

For any $X \subseteq N$, $Y \subseteq N$

$$P_X . P_Y(a) = P_X(P_Y(a))$$

That is, in this example one first operates with P_Y, then with P_X.
DF(10): Square brackets, [ ], around a pair of numbers define a function on $(N \cup \{\phi\})^2$ such that if $x = \phi$ or $y = \phi$ or $x = y = \phi$ then $[x,y] = \phi$. If x, $y \in N$ then $[x,y] \in N$ and if $g^{-1}(x)$, $g^{-1}(y) \in P$ then $g^{-1}[x,y] \in P$.

DF(11): The product is now introduced. For every triple $\langle a,b,c\rangle$ of numbers in N and every quartuple $\langle X, Y, Z, W\rangle$ of sets of numbers in N

$$G_1(P_X(a),P_Y(b)),\ G_1(P_Z(b),P_W(c)) \in E \rightarrow$$
$$G_1([P_X(a),P_Z(b)],\ [P_Y(b),P_W(c)]) \in E$$

$G_1([P_X(a),P_Z(b)],\ [P_Y(b),P_W(c)])$ is called the product of

$G_1(P_X(a),P_Y(b))$ and $G_1(P_Z(b),P_W(c))$.

RMK(5): Consider $G_1([a,P_Z(b)],\ [P_Y(b),c])$. If

$$P_Z\,.\,P_Y(n) = \phi$$

for each $n \in N$,

then

$$G_1([a,P_Z(b)],\ [P_Y(b),c]) = \Phi$$

On the other hand, if Y, Z are sets such that

$$P_Z\,.\,P_Y(b) = b$$

Then

$$G_1([a,b],\ [b,c]) \in E$$

There are several alternative definitions (U(1) to U(4)) which might have been used for the product instead of DF(11). In U(1) to U(4), a, b, $c \in N$ are numbers and X, Y, Z, $W \in N$ are sets of numbers. The expression to the right of the implication sign is the one that might have been used for the product.

U(1):

$$G_1(P_X(a),P_Y(b)),\ G_1(P_Z(c),P_W(d)) \in E \rightarrow$$
$$G_1([P_X(a),P_Z(c)],\ [P_Y(b),P_W(d)]) \in E$$

With $c \neq d$, U(1) leads to nothing beyond the ordinary results for ordered pairs. DF(11) will later be shown to have physical relevance.

U(2):

$$G_1(P_Z(b),P_W(c)),\ G_1(P_X(a),P_Y(b)) \in E \rightarrow$$
$$G_1([P_Z(b),P_X(a)],\ [P_W(c),P_Y(b)]) \in E$$

If used instead of DF(11), the results are equivalent though not identical to those obtained by DF(11), as is also true of U(3) and U(4).

U(3):

$$G_1(P_X(a),P_Y(b)),\ G_1(P_Z(b),P_W(c)) \in E \rightarrow$$
$$G_1([P_X(a),P_Y(b)],\ [P_Z(b),P_W(c)]) \in E$$

U(4):

$$G_1(P_X(a), P_Y(b)),\ G_1(P_Z(b), P_W(c)) \in E \rightarrow$$
$$G_1([P_X(a), P_Z(b)],\ [P_W(c), P_Y(b)]) \in E$$

RMK/DEF(6): Z_8 is described by axioms AX(1) to AX(5) and definitions DF(1) to DF(10) together with axioms AX(6) and AX(7) and definitions DF(12), DF(13), DF(14), DF(15) details of which are given below. From now on Z_8 is the formal system we refer to unless contrary specification be given.
AX(6): Z_8 is simply consistent.
AX(7): Z_8 is symmetric.
RMK(7): AX(7) implies that for each $H \in P$, $n \in N$, if $G(h,n) \in T$ then there exists $H' \in P$ such that $G(h',n) \in R$.

Note also that every Tarski theory is symmetric.
DF(12): Now define a new function P_η on $\{h | g^{-1}(h) \in P\} \cup \{\phi\}$

$$P_\eta(h) = h', \qquad P_\eta(h') = h, \qquad g^{-1}(h),\ g^{-1}(h') \in P$$

The notation used is the same as is used in RMK(7).
DF(13): Define P_A, a function on $N \cup \{\phi\}$

$$P_A(n) = n \qquad \text{if} \qquad g^{-1}(n) \in P \qquad \text{and} \qquad P_A \,.\, P_\eta(n) = \phi$$

Otherwise

$$P_A(n) = \phi \qquad \text{if} \qquad g^{-1}(n) \in P$$

In this case

$$P_A \,.\, P_\eta(n) = P_\eta(n)$$

$$P_A(v) = v \qquad \text{if for some} \qquad g^{-1}(h) \in P, \qquad G(h,v) \in T, \qquad v \in N$$

and

$$P_A(h) = h \qquad \text{and} \qquad G(m,v) \notin R, \qquad g^{-1}(m) \in P; \qquad P_A(m) \neq \phi$$

Otherwise

$$P_A(n) = \phi \qquad n \in N \cup \{\phi\}$$

RMK(8): DF (13) implies that, if the set of predicates in Z_8 which has the set of Godel numbers $A \subset N$ contains an element H where $G(h,n) \in T$ (i.e. $G(h,n)$ is 'true') then it does not contain the element H', $G(h',n) \in R$.
DF(14): The product used in Z_8 is defined as follows. For every triple $\langle a,b,c \rangle$ of numbers in N and every quartuple $\langle \alpha,\beta,\gamma,\delta \rangle$ where P_α, P_β,

P_γ, P_δ are functions defined by CVN(5) or DEF(12) or may be any dot products of both kinds of functions,

$$G_1(P_\alpha(a), P_\beta(b)),\ G_1(P_\gamma(b), P_\delta(c)) \in E \rightarrow$$
$$G_1([P_\alpha(a), P_A . P_\gamma(b)],\ [P_\beta,(b), P_A . P_\delta(c)]) \in E \cup \Phi$$

$G_1([P_\alpha(a), P_A . P_\gamma(b)],\ [P_\beta(b), P_A . P_\delta(c)])$ is said to be the product of the two expressions to the left of the implication sign. In DF(14), $G_1(P_\gamma(b), P_\delta(c))$ is said to right-multiply $G_1(P_\alpha(a), P_\beta(b))$ to give the product. A restriction is made on the binary mapping [ ] in DF(15).

RMK(9): Several results follow immediately in Z_8.

TH(2): $G_1([q,q],[q,q])$ and $G_1([P_\eta(q), P_\eta(q)],\ [P_\eta(q), P_\eta(q)])$, $(q \in N)$ cannot both be contained in E.

TH(3): $G_1([q,q],[q,q])$ and $G_1([P_\eta(q), q], [P_\eta(q), q])$, $(q \in N)$ can both be contained in E.

TH(4): $G_1([q,q],[q,q])$ and $G_1([q, P_\eta(q)], [q, P_\eta(q)])$, $(q \in N)$ cannot both be contained in E.

RMK(10): U(2), U(3), and U(4) clearly cannot be used usefully as co-definitions with DF(14), even if they are restricted in some way by use of P_A.

RMK(11): One consequence of DF(14) is that, if one expression can be right-multiplied by another, it cannot be right-multiplied by the converse of the latter to give an expression in Z_8.

If $G_1([q,q],[q,q]) \in E$, then any expression with $g^{-1}(P_\eta(q))$ as predicate, even including $G_1(P_\eta(q), P_\eta(q))$ would not give an expression in Z_8 on right-multiplication. In part II, we deal with systems where, inevitably, for some $q \in N$, $G_1(P_\eta(q), P_\eta(q)) \in T$ holds. It will be there seen that the calculus is usefully simplified by the restrictions of DF(14) beyond those which DF(11) would produce.

RMK(12): Consider

$$\{P_A(a) | g^{-1}(a) \in P\} = T_0$$
$$\{P_\eta . P_A(a) | g^{-1}(a) \in P\} = R_0$$

Then for each $r^{(1)}$, $r^{(2)} \in R_0$, $t^{(1)}$, $t^{(2)} \in T_0$

$$[r^{(1)}, P_A(r^{(2)})] = [t^{(1)}, P_A(r^{(2)})] = \phi$$

Also, for some $t^{(3)}$, $t^{(4)} \in T_0$, $r^{(3)}$, $r^{(4)} \in R_0$

$$[r^{(1)}, P_A(t^{(2)})] = t^{(3)} \quad \text{or} \quad [r^{(1)}, P_A(t^{(2)})] = r^{(3)}$$
$$[t^{(1)}, P_A(t^{(2)})] = t^{(4)} \quad \text{or} \quad [t^{(1)}, P_A(t^{(2)})] = r^{(4)}$$

If product formation is associative, as we will require, then

$$[t^{(1)}, P_A[r^{(1)}, P_A(t^{(2)})]] = [[t^{(1)}, P_A(r^{(1)})], P_A(t^{(2)})]$$
$$= [\phi, t^{(2)}] = \phi$$
$$[r^{(1)}, t^{(2)}] = r^{(3)}$$

Also

$$[t^{(1)}, t^{(2)}] = t^{(4)}$$

Complete the definition of Z_8 by adding DF(15).
DF(15): For each $a, b \in N$

$$P_A([a, P_A(b)]) \neq \phi \leftrightarrow P_A(a) = a, P_A(b) = b$$
$$P_A . P_\eta([a, P_A(b)]) \neq \phi \leftrightarrow P_A . P_\eta(a) = a, P_A(b) = b$$

CVN(6): $[a, P_A(b)] = [a, b]'$ for each $a, b \in N$.
RMK(13): Z_8 is used in Part II to facilitate the definition of determinism of physical processes.

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